Sec 4.4, 4.5: Homogeneous Systems with Constant Coefficients

Has the standard form

$$\vec{Y}\,' = A \cdot \vec{Y}, \qquad -\infty < t < \infty,$$

where A is a constant matrix.

Thm 1. If A ($n \times n$ matrix) has n distinct eigenvalues, say $\lambda_1, \lambda_2, \cdots, \lambda_n$, then the matrix

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\vec{v}_1 & e^{\lambda_2 t}\vec{v}_2 & \cdots & e^{\lambda_n t}\vec{v}_n \end{bmatrix},$$

where $\vec{v}_i$ is an eigenvector associated to λ_i, is a fundamental matrix for $\vec{Y}\,' = A \cdot \vec{Y}$.

TO DO:

Ex1. Give the general solution to the homogeneous equation: $\vec{Y}\,' = A \cdot \vec{Y}$, where $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$.

① Find Eigenpairs → i) Eigenvalues: $P(\lambda) = \det[A - \lambda I] = 0$

$\Leftrightarrow \det\left[\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

$P(\lambda) = \det\begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix} = (4-\lambda)(1-\lambda)+2$

$= 4 - \lambda - 4\lambda + \lambda^2 + 2 \Rightarrow P(\lambda) = \lambda^2 - 5\lambda + 6$

$= (\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda_1 = 2, \ \lambda_2 = 3$

② For each Eigenvalue find the associated Eigenvector

for $\lambda_1 = 2$ Solve for the associated eigenvector by solving

$(A - 2I)\vec{v} = \vec{0} \in \mathbb{R}^2$

$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{v_1 - v_2 = 0} \Rightarrow v_1 = v_2$ or is the span $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eigenvector associated with $\lambda_1 = 2$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow$ Eigenpairs $\left(2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

$\Rightarrow$ one column of any Fund matrix $\Phi_1(t) = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\lambda_2 = 3$ look for the associated eigenvector by solving $(A - 3I)\vec{v} = \vec{0} \in \mathbb{R}^2$

$\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 - 2v_2 = 0 \qquad v_1 = 2v_2 \Rightarrow \begin{pmatrix} 2v_2 \\ v_2 \end{pmatrix} = v_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Eigenpair $\left(3, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) \Rightarrow \Phi_2(t) = e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

③ Build the solution

$$\Phi(t) = \left[e^{2t}\begin{pmatrix}1\\1\end{pmatrix} \quad e^{3t}\begin{pmatrix}2\\1\end{pmatrix} \right]$$

and the general solution $\vec{y}(t) = \Phi(t)\begin{pmatrix}c_1\\c_2\end{pmatrix}$

$$\vec{y}(t) = c_1 e^{2t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{3t}\begin{pmatrix}2\\1\end{pmatrix}$$

Thm 2 [Recognizing a Fundamental Matrix] Suppose A ($n \times n$ matrix) has a full set of eigenvecto

That is, if

$$\begin{array}{lll}
\lambda_\mathbf{1} \quad \text{provides:} & \vec{v}_{\mathbf{1},1},\ \vec{v}_{\mathbf{1},2},\ \vec{v}_{\mathbf{1},3}, \cdots \vec{v}_{\mathbf{1},r_1} & \text{linearly independent eigenvector(s)} \\
\lambda_\mathbf{2} \quad \text{provides:} & \vec{v}_{\mathbf{2},1},\ \vec{v}_{\mathbf{2},2},\ \vec{v}_{\mathbf{2},3}, \cdots \vec{v}_{\mathbf{2},r_2} & \text{linearly independent eigenvector(s)} \\
 & \cdots & \\
\lambda_\mathbf{k} \quad \text{provides:} & \vec{v}_{\mathbf{k},1},\ \vec{v}_{\mathbf{k},2},\ \vec{v}_{\mathbf{k},3}, \cdots \vec{v}_{\mathbf{k},r_k} & \text{linearly independent eigenvector(s),}
\end{array}$$

where $r_i = \text{GM}(\lambda_i) = \text{AM}(\lambda_i)$ for each $i = 1, 2, \ldots, k$. Then, the matrix

$$\Phi(t) = \left[e^{\lambda_1 t}\vec{v}_{\mathbf{1},1} \quad e^{\lambda_1 t}\vec{v}_{\mathbf{1},2} \quad \cdots \quad e^{\lambda_1 t}\vec{v}_{\mathbf{1},r_1} \quad \cdots\cdots \quad e^{\lambda_k t}\vec{v}_{\mathbf{k},1} \quad e^{\lambda_k t}\vec{v}_{\mathbf{k},2} \quad \cdots \quad e^{\lambda_k t}\vec{v}_{\mathbf{k},r_k} \right]$$

is a fundamental matrix for $\vec{Y}\,' = A \cdot \vec{Y}$.

Important remarks:

- $\mathbf{k}$ is the number of distinct eigenvalues of the matrix A. The number $\mathbf{k}$ may not be n.
- Theorem 1 is the particular case when $\mathbf{k} = n$.